On the asymptotic reduction to the multidimensional nonlinear Schrodinger equation [1]

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Abstract

The problem on the asymptotics for the solution of multidimensional nonlinear Boussinesq equation with respect to a small parameter ε is considered. The asymptotic expansion of the solution of this problem with respect to $\varepsilon \to 0$ for long times $t \sim \mathcal{O}(\varepsilon^{-2})$ is constructed and justified. The leading terms of the asymptotic solution are defined from the multidimensional nonlinear Schrodinger equation and from the linear homogeneous wave equation.

1 Introduction

In this work the problem on the asymptotic reduction in multidimensional waves is investigated where the construction of the asymptotic solution reduces to the multidimensional nonlinear Schrodinger equation.

The problem on the asymptotics with respect to the small parameter ε for the multidimensional nonlinear Boussinesq equation is considered:

$$\partial_t^2 u - \Delta_{x\bar{y}} u + \nu[\partial_x^4 + \partial_{\bar{y}}^4]u + \operatorname{div} \mathbf{g}(\nabla u) = 0, \quad x \in \mathbb{R}, \ \bar{y} \in \mathbb{R}^{n-1}, \ t > 0, \tag{1}$$

Here $\nabla = (\partial_x, \nabla_{\bar{y}}),\ \nabla_{\bar{y}} = (\partial_{y_2}, \dots, \partial_{y_n}),\ \Delta_{\bar{y}} = \nabla_{\bar{y}}^2,\ \Delta_{x\bar{y}} = \partial_x^2 + \Delta_{\bar{y}},\ \partial_{\bar{y}}^4 = \partial_{y_2}^4 + \ldots + \partial_{y_n}^4,\ \operatorname{div} \mathbf{g} =< \nabla \cdot \mathbf{g} >,\ \nu = \text{const} > 0$. The initial conditions are taken as a combination of plane waves with small amplitudes:

$$\begin{bmatrix} u \\ u_t \end{bmatrix} (x, \bar{y}, t, \varepsilon)_{|t=0} = \varepsilon \sum_{k=0,\pm 1} \Lambda_k e^{ikx}, \tag{2}$$

and deformed by slow variables:

$$\Lambda_k = \begin{bmatrix} \varphi_k \\ \psi_k \end{bmatrix} (\varepsilon x, \varepsilon \bar{y}), \quad \Lambda_k^* = \Lambda_{-k}, \quad \psi_0 \equiv 0.$$

Their components are considered as functions which decrease fast at infinity:

$$\varphi_k, \psi_k(\xi, \bar{\eta}) = \mathcal{O}((|\xi| + |\bar{\eta}|)^{-N}), \quad |\xi| + |\bar{\eta}| \to \infty, \quad \forall N > 0, \quad \forall k. \tag{3}$$

The purpose of this work is the construction and justification of asymptotic expansion of the solution $u(x, \bar{y}, t, \varepsilon)$ of the Cauchy problem (1)–(2) as $\varepsilon \to 0$, uniformly on a long-time interval $0 \le t \le \mathcal{O}(\varepsilon^{-2}),\ \forall (x, \bar{y}) \in \mathbb{R}^n$.

The main result is the following. The determination of leading terms of the asymptotic solution of the problem (1)–(2) is reduced to the solution of the Cauchy problems for the multidimensional nonlinear Schrodinger equation (NLS):

$$i\partial_\theta w + \alpha \partial_\sigma^2 w + \delta \Delta_{\bar{\eta}} w + \gamma w |w|^2 = 0, \quad \theta > 0, \tag{4}$$

$$w(\sigma, \bar{\eta}, \theta)_{|\theta=0} = w_0(\sigma, \bar{\eta}), \quad (\sigma, \bar{\eta}) \in \mathbb{R}^n, \tag{5}$$

and for the linear homogeneous wave equation:

$$\partial_\tau^2 \phi - \Delta_{\xi\bar{\eta}} \phi = 0, \ \tau > 0; \quad \phi_{|\tau=0} = \varphi_0(\xi, \bar{\eta}), \quad \partial_\tau \phi_{|\tau=0} = 0, \quad (\xi, \bar{\eta}) \in \mathbb{R}^n. \tag{6}$$

The formal constructions of the asymptotic solution of the Cauchy problem (1)–(2) are obtained here without any essential restrictions of the input data. We only suppose that the components of vector

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function $\mathbf{g}$ are decomposed into the asymptotic Taylor serieses as $|v|+|\bar{w}| \to 0$ ($v \in \mathbb{R},\ \bar{w} \in \mathbb{R}^{n-1}$) with specific quadratic terms: [2]

$$g_j(v,\bar{w}) = g^{0,2}_{j,i} w_i^2 + g^{3,0}_j v^3 + g^{2,1}_{j,i} v^2 w_i + g^{1,2}_{j,i} v w_i^2 + g^{0,3}_{j,i} w_i^3 + \ldots,\ j = \overline{1,n}. \tag{7}$$

It is worth noting that in case of general nonlinearity the formal constructions reduce to the Davey-Stewartson type systems of equations [1].

Justification of the asymptotic expansion will be carried out under the side conditions which are connected with the proof of solvability of the input problem. One of the requirements is the absence of the part of the cubic terms in expansion (7):

$$g^{3,0}_j \equiv 0, \quad j = \overline{2,n}. \tag{8}$$

For the two-dimensional case ($n = 2$) the formal asymptotic reductions to the NLS and Davey-Stewartson type systems of equations are known from [2, 3, 4]. Justification of these asymptoticses for multidimensional problems was obtained only by few authors. For example, justification of reduction to the "shallow water" equations for 2+1-dimensional surface waves in a class of analytical functions is obtained in [5]; justification of reduction to the Kadomtsev-Petviashvili equation is reduced in [1].

In the present work we justify the asymptotic reduction to the multidimensional NLS. The main mathematical result consists of solvability of the input problem and evaluations of the residual of the asymptotic expansion. The obtained result gives the exact sense of asymptotic reduction to the multidimensional NLS. It is worth mentioning that here we use the ideas and sentences expressed in [1, 6].

The problem (1)–(2) is considered as a simplest example where the construction of asymptotic solution reduces to the multidimensional NLS. Similar results may be obtained and for the more complicated equations of type (1), generally, for pseudodifferential ones.

Here we shall be limited to reviewing of the equation (1) for revealing basic singularities of the construction and justification of the asymptotic expansion in multidimensional problems.

Theorem 1. *Let's suppose the functions $g_j(\nabla u)$, $j = \overline{1,n}$ are analytical in neighbourhood of zero so Taylor serieses (7), (8) converge at $|\nabla u| \le M$. Initial functions $\varphi_k, \psi_k(\xi,\bar{\eta})$, $k = 0, \pm 1$ satisfy to (3). Let them be analytical in layer $S(\beta_0) = \{|Im\xi|, |Im\eta_2|, \ldots, |Im\eta_n| \le \beta_0\}$ $(\beta_0 = const > 0)$. Then there are the values $\varepsilon_0, T > 0$, $\beta \in (0,\beta_0)$: $\forall \varepsilon \in (0,\varepsilon_0)$ the Cauchy problem (1)–(2) has a unique solution $u(x,\bar{y},t,\varepsilon)$ in $\Omega(T) = \{(x,\bar{y}) \in \mathbb{R}^n, 0 \le t \le T\varepsilon^{-2}\}$ which is analytical in layer $S(\beta)$. This solution has the following asymptotics:*

$$u = \varepsilon\Big[v_0(\xi,\bar{\eta},\tau) + \sum_{k=\pm 1}\sum_{\omega=\pm\omega(k)} w_{k,\omega}(\sigma,\bar{\eta},\theta)e^{ikx-i\omega t}\Big] + \mathcal{O}(\varepsilon^2),$$

$$\xi = \varepsilon x,\ \bar{\eta} = \varepsilon\bar{y},\ \tau = \varepsilon t,\ \theta = \varepsilon^2 t,\ \sigma = \xi - \omega'\tau,\ \omega(k) = k\sqrt{1+\nu k^2},$$

uniformly in $\Omega(T)$. The complex amplitudes $w = w_{k,\omega}$ are the solutions of the Cauchy problem (4)–(5) for the nonlinear Schrodinger equations with constants: $\alpha = \omega''/2$, $\delta = 1/(2\omega)$, $\gamma = 3g^{3,0}_1/(2\omega)$ and with initial value: $w_0 = 1/2\varphi_k + i\delta\psi_k$. The amplitude of zero harmonic $\phi = v_0$ is defined from the Cauchy problem (6) for the homogeneous wave equation.

The functions $w_{k,\omega}(\sigma,\bar{\eta},\theta)$ represent slowly deformed (in a scale $\theta = \varepsilon^2 t$) amplitudes of wave packets travelling with group velocity ω' in characteristic directions $x - \omega' t = $ const. These amplitudes are modulated both in longitudinal ($\sigma = \varepsilon(x - \omega' t)$) and in cross ($\bar{\eta} = \varepsilon\bar{y}$) directions.

2 The formal constructions

For the formal construction of the asymptotic solution of the problem (1)–(2) we shall use the method of multiply scale [7]. For this purpose we use the above entered slow variables $\xi, \bar{\eta}, \tau, \theta$. We shall search for a solution in the form of $u(x,\bar{y},t,\varepsilon) = v(x,\xi,\bar{\eta},t,\tau,\theta,\varepsilon)$. Taking into account the fact that derivatives will be transformed by the rules:

$$\begin{aligned} \partial_t u &= \partial_t v + \varepsilon\partial_\tau v + \varepsilon^2\partial_\theta v \equiv D_t v, \\ \partial_x u &= \partial_x v + \varepsilon\partial_\xi v \equiv D_x v, \\ \nabla_{\bar{y}} u &= \varepsilon\nabla_{\bar{\eta}} v, \end{aligned}$$

[2] Here summation is conducted along coinciding indexes

we shall obtain the following equation for v:

$$\begin{aligned}
[\partial_t^2 - \partial_x^2 + \nu\partial_x^4]v &+ 2\varepsilon[\partial_t\partial_\tau - \partial_x\partial_\xi + 2\nu\partial_x^3\partial_\xi]v + \\
&+ \varepsilon^2[\partial_\tau^2 + 2\partial_t\partial_\theta - \partial_\xi^2 - \Delta_{\bar\eta} + 6\nu\partial_x^2\partial_\xi^2]v + \\
&+ 2\varepsilon^3[\partial_\tau\partial_\theta + 2\nu\partial_x\partial_\xi^3]v + \varepsilon^4[\partial_\theta^2 + \nu\partial_\xi^4 + \nu\partial_{\bar\eta}^4]v + \\
&+ D_x g_1(D_x v, \varepsilon\nabla_{\bar\eta} v) + \varepsilon < \nabla_{\bar\eta}\cdot \bar g(D_x v, \varepsilon\nabla_{\bar\eta} v) >= 0 \qquad (9)
\end{aligned}$$

with the initial conditions:

$$\begin{bmatrix} v \\ D_t v \end{bmatrix}_{|t=\tau=\theta=0} = \varepsilon \sum_{k=0,\pm1} \Lambda_k(\xi,\bar\eta)e^{ikx}. \qquad (10)$$

The *formal asymptotic solution* (FAS) of the Cauchy problem (9)–(10) is called a segment of series

$$v \sim \varepsilon \sum_{n=0}^{\infty} \varepsilon^n \overset{n}{v} (x,t,\xi,\bar\eta,\tau,\theta), \qquad (11)$$

which, first of all, satisfies to equation (9) with exactitude $\mathcal{O}(\varepsilon^4)$ and to the initial values (10) with exactitude $\mathcal{O}(\varepsilon^3)$ and, secondly, each consequent term of this segment is less than the previous one in an ε order, uniformly in the corresponding domain with respect to independent variables.

By the method explained below we may construct any length of the series's segment (11), sequentially defining dependence of coefficients of asymptotics in all temporal scales. Only three terms of this series are constructed here. It is enough for deriving the required residuals. The leading term $\overset{0}{v}$ of the asymptotics is finally defined dependending on variables t,τ,θ; the first correction $\overset{1}{v}$ is completely defined in scales t,τ, the second correction $\overset{2}{v}$ - in scale t.

The coefficients $\overset{n}{v}$ are constructed in the form of final Fourier sums:

$$\overset{n}{v} = \sum_{k,\omega} \overset{n}{v}_{k,\omega} (\xi,\bar\eta,\tau,\theta)e^{ikx-i\omega t}, \quad (\overset{n}{v}_{k,\omega} = \overset{n}{v}{}^{*}_{-k,-\omega}), \qquad (12)$$

where sign $*$ denotes complex conjugate. The problem is reduced to defining the coefficients $\overset{n}{v}_{k,\omega}$.

Substitutingserieses (11)–(12) into the equation (9) and equating terms at identical degrees of parameter ε and identical indexes k,ω, we shall obtain the equations on $\overset{n}{v}_{k,\omega}$. So, on the first step, at ε^1 we obtain the homogeneous equations on $\overset{0}{v}$:

$$[\partial_t^2 - \partial_x^2 + \nu\partial_x^4] \overset{0}{v} = 0.$$

Dispersion relation is defined from here:

$$\omega^2 - k^2 - \nu k^4 = 0,$$

whence $\omega = \pm\omega(k) \equiv \pm k\sqrt{1+\nu k^2}$. The values $k = 0, \pm1$ are taken according to the initial data (10). Hence, we have the following representation for $\overset{0}{v}$:

$$\overset{0}{v} = v_0(\xi,\bar\eta,\tau,\theta) + \sum_{k=\pm1}\sum_{\omega=\pm\omega(k)} \overset{0}{v}_{k,\omega} (\xi,\bar\eta,\tau,\theta)e^{ikx-i\omega t}.$$

The initial conditions (10) give values of the functions v_0 and $\overset{0}{v}_{k,\omega}$ at $\tau = \theta = 0$:

$$v_{0|\tau=\theta=0} = \varphi_0(\xi,\bar\eta),$$

$$\begin{aligned}
&[\overset{0}{v}_{k,\omega(k)} + \overset{0}{v}_{k,-\omega(k)}]_{|\tau=\theta=0} = \varphi_k(\xi,\bar\eta), \\
-i\omega(k) \quad &[\overset{0}{v}_{k,\omega(k)} - \overset{0}{v}_{k,-\omega(k)}]_{|\tau=\theta=0} = \psi_k(\xi,\bar\eta), \quad k=\pm1.
\end{aligned} \qquad (13)$$

On the second step, equating expressions at ε^2, we shall obtain the inhomogeneous equation:

$$-[\partial_t^2 - \partial_x^2 + \nu\partial_x^4] \overset{1}{v} = 2[\partial_t\partial_\tau - \partial_x\partial_\xi + 2\nu\partial_x^3\partial_\xi] \overset{0}{v} . \qquad (14)$$

In order that solution $\overset{1}{v}$ shouldn't have secular (growing at $t \to \infty$) terms it is necessary to equate to zero the functions in right-hand side of (14) which are the solutions of the corresponding homogeneous equation. Whereas

$$[\partial_t\partial_\tau - \partial_x\partial_\xi + 2\nu\partial_x^3\partial_\xi]\,\overset{0}{v} = -\sum_{k=\pm1}\sum_{\omega=\pm\omega(k)} i\omega e^{ikx-i\omega t}[\partial_\tau + \omega^{'}(k)\partial_\xi]\,\overset{0}{v}_{k,\omega},$$

the right-hand side of (14) consists completely of solutions of the homogeneous equation. Therefore they should be excluded from the requirements:

$$[\partial_\tau + \omega^{'}(k)\partial_\xi]\,\overset{0}{v}_{k,\omega} = 0, \quad k=1,\ \omega=\pm\omega(k), \quad \xi\in\mathbb{R},\ \tau>0. \tag{15}$$

These equations are trivial and allow in functions $\overset{0}{v}_{k,\omega},\ \forall k\neq 0$ to define a structure of dependence from ξ,τ (on the first slow scale) for all values $\tau\geq 0$:

$$\overset{0}{v}_{k,\omega}\,(\xi,\bar{\eta},\tau,\theta) = w_{k,\omega}(\sigma_{k,\omega},\bar{\eta},\theta), \quad \sigma_{k,\omega} = \xi - \omega^{'}\tau, \quad k=1,\ \omega=\pm\omega(k).$$

Thus the number of independent variables is diminished in the coefficients of the leading term of the asymptotics. The initial conditions (13) define functions $w_{k,\omega}$ at $\theta=0$:

$$w_{k,\omega}(\sigma_{k,\omega},\bar{\eta},\theta)_{|\theta=0} = \frac{1}{2}\left(\varphi_k + \frac{i}{\omega}\psi_k\right)(\sigma_{k,\omega},\bar{\eta}), \quad k=1,\ \omega=\pm\omega(k).$$

The values of these functions as well as the leading term of the asymptotics remain undefined from slow dependence on time θ (in the second scale). In order to define dependence on θ it is necessary to analyze the next corrections of the asymptotic solution.

After the carried out elemination of the secular terms first the correction satisfies to the equation:

$$[\partial_t^2 - \partial_x^2 + \nu\partial_x^4]\,\overset{1}{v} = 0. \tag{16}$$

The particular solution of the equation (16) is trivial: $\overset{1}{v}=0$.

The general solution of the equation (16) is supplemented by the solution of the homogeneous one:

$$\overset{1}{v} = \sum_{k=\pm1,\pm2}\sum_{\omega=\pm\omega(k)} \overset{1}{v}_{k,\omega}\,(\xi,\bar{\eta},\tau,\theta)e^{ikx-i\omega t}$$

with undefined coefficients $\overset{1}{v}_{k,\omega}$ for the time being.

The initial conditions (10) in an ε^2 order give initial values for these functions and also for $\partial_\tau v_0$:

for $k=0$:

$$\partial_\tau v_0(\xi,\bar{\eta},\tau,\theta)_{|\tau=\theta=0} = 0;$$

for $k=\pm1$:

$$\begin{array}{rl} & [\overset{1}{v}_{k,\omega(k)} + \overset{1}{v}_{k,-\omega(k)}]_{|\tau=\theta=0} = 0, \\ -i\omega(k) & [\overset{1}{v}_{k,\omega(k)} - \overset{1}{v}_{k,-\omega(k)}]_{|\tau=\theta=0} = -\partial_\tau[\overset{0}{v}_{k,\omega(k)} + \overset{0}{v}_{k,-\omega(k)}]_{|\tau=\theta=0}; \end{array} \tag{17}$$

for $k=\pm2$:

$$\begin{array}{rl} & [\overset{1}{v}_{k,\omega(k)} + \overset{1}{v}_{k,-\omega(k)}]_{|\tau=\theta=0} = 0, \\ -i\omega(k) & [\overset{1}{v}_{k,\omega(k)} - \overset{1}{v}_{k,-\omega(k)}]_{|\tau=\theta=0} = 0. \end{array} \tag{18}$$

The amplitudes $\overset{1}{v}_{k,\omega(k)},\ k=1,2,\ \omega=\pm\omega(k)$, on this step remain undefined in dependence on $\tau,\ \theta>0$. The amplitude of zero harmonic v_0 remains undefined too. The equations for them will be obtained on the following step.

In an ε^3 order we obtain a linear inhomogeneous equation:

$$\begin{aligned} -[\partial_t^2 - \partial_x^2 + \nu\partial_x^4]\,\overset{2}{v} &= 2[\partial_t\partial_\tau - \partial_x\partial_\xi + 2\nu\partial_x^3\partial_\xi]\,\overset{1}{v} + \\ &+ [\partial_\tau^2 - \partial_\xi^2 - \Delta_{\bar{\eta}} + 2\partial_t\partial_\theta + 6\nu\partial_x^2\partial_\xi^2]\,\overset{0}{v} + \\ &+ g_1^{3,0}\partial_x(\partial_x\,\overset{0}{v})^3, \end{aligned} \tag{19}$$

from the right-hand side of which it is necessary to eliminate secular addends. The elemination of secularities reduces to the equations for amplitudes $\overset{1}{v}_{k,\omega},\ k=1,\ \omega=\pm\omega(k)$ of the first correction in scale ξ,τ :

$$2i\omega[\partial_\tau+\omega^{'}(k)\partial_\xi]\overset{1}{v}_{k,\omega}=[-2i\omega\partial_\theta+((\omega^{'})^2-1-6\nu k^2)\partial_\xi^2-\Delta_{\bar\eta}]w_{k,\omega}- \\ -3g_1^{3,0}w_{k,\omega}[|w_{k,\omega}|^2+2|w_{k,-\omega}|^2],\quad k=1,\ \omega=\pm\omega(k). \tag{20}$$

On the same step we obtain the equations for the amplitude of zero harmonic of the leading term:

$$\partial_\tau^2 v_0-\Delta_{\xi\bar\eta}v_0=0. \tag{21}$$

We have obtained the initial conditions for v_0 on the previous steps, namely:

$$\left[\begin{array}{c} v_0 \\ \partial_\tau v_0 \end{array}\right]_{|\tau=\theta=0}=\left[\begin{array}{c}\varphi_0(\xi,\bar\eta)\\ 0\end{array}\right],\quad (\xi,\bar\eta)\in\mathbb{R}^{\,n}. \tag{22}$$

So, on the first slow scale τ 0 the amplitudes of the leading term of the asymptotics are defined from the equations (15), (21). These equations are linear. Just this circumstance allows "to attain" up to long times $t\approx\varepsilon^{-2}$ in formal solution. To guarantee the absence of secularities in first two orders of the formal solution we should analyze the equations (20), (21) their solutions' boundedness at $\tau\to\infty$.

It is known in case of rapidly decreasing at infinity of the initial data (22), the solution of the Cauchy problem for two-dimensional ($n=2$) wave equation (21) decreases at $\tau\to\infty$ [8]. The result of the uniformly boundedness of the solution of the Cauchy problem (21)–(22) for the case of multidimensions will be represented in the following section.

Let's pass to the equations (20). Analyzing their solutions we obtain the equations for $w_{k,\omega}(\sigma_{k,\omega},\bar\eta,\theta)$ ($\sigma_{k,\omega}=\xi-\omega^{'}\tau, k=1,\ \omega=\pm\omega(k)$). The right-hand side of the equations (20) contains the solutions of homogeneous one as functions depending on $\sigma=\sigma_{k,\omega}=\xi-\omega^{'}\tau$. In order that solutions of inhomogeneous equations (20) shouldn't contain secular terms we must eliminate the above mentioned functions. Equating to zero of expressions, depending on $\sigma=\sigma_{k,\omega}$ reduces to nonlinear Schrodinger equations ($k=1,\ \omega=\pm\omega(k)$) :

$$[-2i\omega\partial_\theta+((\omega^{'})^2-1-6\nu k^2)\partial_\sigma^2-\Delta_{\bar\eta}]w_{k,\omega}=3g_1^{3,0}w_{k,\omega}|w_{k,\omega}|^2. \tag{23}$$

So, in characteristic directions $\sigma=\sigma_{k,\omega}=\xi-\omega^{'}\tau=\text{const}$ at $|\xi|+\tau\to\infty$ the amplitudes of the leading term of the formal solution are defined from (21) and (23). According to

$$\frac{((\omega^{'})^2-1-6\nu k^2)}{-\omega}=\frac{1+6\nu k^2}{\omega}-\frac{(k+2\nu k^3)^2}{\omega^3}=\frac{d^2\omega}{dk^2}\equiv\omega^{''},$$

the equation (23) for $w=w_{k,\omega}$ has the form:

$$i\partial_\theta w+\alpha\partial_\sigma^2 w+\delta\Delta_{\bar\eta}w+\gamma w|w|^2=0, \tag{24}$$

where $\alpha=\omega^{"}/2,\ \delta=1/(2\omega),\ \gamma=3g_1^{3,0}/(2\omega)$. The equation (24) describes the slow (in scale $\theta=\varepsilon^2 t$) deformation of the amplitudes of wave packages.

The equation (24) is supplemented by the initial condition:

$$w(\sigma,\bar\eta,\theta)_{|\theta=0}=w_0(\sigma,\bar\eta), \tag{25}$$

where $w_0=1/2\varphi_k+i\delta\psi_k$.

The solvability of the problem (24)–(25) in a class of functions decreasing at $|\sigma|+|\bar\eta|\to\infty$ can be proved similarly [9, 10]. This result will be represented in the next section. It guarantees decreasing on infinity with respect to $\xi,\bar\eta$ of the right-hand sides of the equations for $\overset{1}{v}_{k,\omega},\ k=1,\ \omega=\pm\omega(k)$:

$$2i\omega[\partial_\tau+\omega^{'}\partial_\xi]\overset{1}{v}_{k,\omega}=-6g_1^{3,0}w_{k,\omega}|w_{k,-\omega}|^2. \tag{26}$$

Eliminating from the right-hand side of (19) of the solutions of the corresponding homogeneous equation at $k=2,\ \omega=\pm\omega(k)$ we shall obtain the equations for $\overset{1}{v}_{k,\omega}$:

$$2i\omega[\partial_\tau+\omega^{'}\partial_\xi]\overset{1}{v}_{k,\omega}=0,\quad k=2,\ \omega=\pm\omega(k). \tag{27}$$

The equations (26), (27) are supplemented by initial conditions, which are obtained from (17), (18) accordingly:

$$\begin{bmatrix} \overset{1}{v}_{k,\omega(k)} \\ \overset{1}{v}_{k,-\omega(k)} \end{bmatrix} (\xi,\bar{\eta},\tau,\theta)_{|\tau=\theta=0} = \begin{bmatrix} \varphi_+ \\ \varphi_- \end{bmatrix} (\xi,\bar{\eta}), \quad k=\pm 1, \tag{28}$$

$$\begin{bmatrix} \overset{1}{v}_{k,\omega(k)} \\ \overset{1}{v}_{k,-\omega(k)} \end{bmatrix} (\xi,\bar{\eta},\tau,\theta)_{|\tau=\theta=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad k=\pm 2, \tag{29}$$

$$\text{where} \quad \varphi_\pm = \mp \frac{1}{2\omega(k)} \partial_\tau [\overset{0}{v}_{k,\omega(k)} + \overset{0}{v}_{k,-\omega(k)}](\xi,\bar{\eta},\tau,\theta)_{|\tau=\theta=0}.$$

The Cauchy problem (27), (29) for $\overset{1}{v}_{k,\omega},\ k=2,\ \omega=\pm\omega(k)$ has only a trivial solution.

The solution $\overset{1}{v}_{k,\omega}=\overset{1}{v}_{k,\omega}\ (\sigma_\pm,\bar{\eta},\tau,\theta),\ k=1,\ \omega=\pm\omega(k)$ of (26) with the initial condition (28) is written out in an explicit form:

$$\overset{1}{v}_{k,\omega}= \varphi_\pm(\sigma_\pm,\bar{\eta}) + 2i\delta w_{k,\omega}(\sigma_\pm,\bar{\eta},\theta) \int\limits_0^\tau |w_{k,-\omega}|^2(\sigma_\pm \pm 2\omega' \mu,\bar{\eta},\theta)d\mu,$$

which is uniformly bounded with respect to τ.

The functions $\overset{1}{v}_{k,\omega},\ k=1,2,\ \omega=\pm\omega(k)$, are defined on this step with exactitude up to the solutions of the corresponding homogeneous equations (26), (27). Dependence of these functions from the second slow scale θ can be defined on the following steps. But it is not necessary for formal constructions which are given here.

After the carried out elemination of the secular terms the second correction satisfies to the equation:

$$\begin{aligned}[\partial_t^2 - \partial_x^2 + \nu\partial_x^4]\, \overset{2}{v}= &\sum_{k=\pm 3} \sum_{\omega=\pm\omega(1);\pm 3\omega(1)} f_{k,\omega}(\xi,\bar{\eta},\tau,\theta)e^{ikx-i\omega t}+ \\ &+\sum_{k=\pm 1} \sum_{\omega=\pm 3\omega(1)} f_{k,\omega}(\xi,\bar{\eta},\tau,\theta)e^{ikx-i\omega t},\end{aligned} \tag{30}$$

$$\text{where} \quad f_{k,\omega} = \begin{cases} \mu(2-k)(w_{k,\bar{\omega}})^2 w_{-k,\bar{\omega}}, & k=\pm 1,\ \bar{\omega}=\pm\omega(1),\ \omega=3\bar{\omega}; \\ -k\mu(w_{k/3,\bar{\omega}})^3, & k=\pm 3,\ \bar{\omega}=\pm\omega(1),\ \omega=3\bar{\omega}; \\ -k\mu(w_{k/3,\omega})^2 w_{k/3,-\omega}, & k=\pm 3,\ \omega=\pm\omega(1). \end{cases}$$

The particular solution of (30) is constructed by Fourier method:

$$\begin{aligned}\overset{2}{v} \quad = &\sum_{k=\pm 3} \sum_{\omega=\pm\omega(1);\pm 3\omega(1)} [-\omega^2+k^2+\nu k^4]^{-1} f_{k,\omega}(\xi,\bar{\eta},\tau,\theta)e^{ikx-i\omega t} + \\ + &\sum_{k=\pm 1} \sum_{\omega=\pm 3\omega(1)} [-\omega^2+k^2+\nu k^4]^{-1} f_{k,\omega}(\xi,\bar{\eta},\tau,\theta)e^{ikx-i\omega t}.\end{aligned}$$

Thus we constructed three terms of FAS of the Cauchy problem (1)–(2) at $\varepsilon \to 0$ for long times $0 \le t \le \mathcal{O}(\varepsilon^{-2}),\ \forall(x,\bar{y}) \in \mathbb{R}^n$. The amplitudes of the leading term of the asymptotics are taken from (24) and (21) which are supplemented by the initial conditions (25) and (22) accordingly.

As we mentioned before, the construction of FAS can be continued for deriving higher corrections $\overset{q}{v},\ q \ge 3$. Here we constructed three terms because it is sufficient for justification of the leading term of the asymptotic expansion.

3 The investigation of standard problems

It is known that the existence of the solution of the Cauchy problem for the multidimensional NLS in Sobolev spaces $H^s(\mathbb{R}^n)$ can be proved by different methods [11, 12]. However we can't obtain justification of the leading term of the asymptotics in $H^s(\mathbb{R}^n)$. Here justification is obtained with severe constraints, namely, with analyticity of input data with respect to spatial variables.

The results of this section of solvability of standard problems (24)–(25) and (21)–(22) represent the generalization of the Cauchy-Kovalevskoi theorem in Ovsyannikov's style [5].

Definition 1. Here we introduce Banach spaces $\mathcal{P}_{\beta,p}$ of functions $U(m,\bar{l})$ with the finite norm

$$\| U \|_p = \sup_{m,\bar{l}} \Big[(1+|m|+|\bar{l}|)^p e^{\beta(1+|m|+|\bar{l}|)} |U(m,\bar{l})| \Big], \quad p \geq (n+1),\ \beta > 0;$$

capital letters are used for Fourier-images:

$$U(m,\bar{l}) = \int\limits_{\mathbb{R}^n} u(\xi,\bar{\eta}) e^{-i(m\xi + <\bar{l}\cdot\bar{\eta}>)} d\xi d\bar{\eta}.$$

It is necessary to notice, that similar spaces with exponential weights in Fourier pre-images correspond to analytical functions in layer $S(\beta) = \{|Im\xi|, |Im\eta_2|, \dots, |Im\eta_n| \leq \beta\}$ [5].

The solution of the Cauchy problem (21)–(22) can be written out in an explicit form:

$$v_0(\xi,\bar{\eta},\tau) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \Phi_0(m,\bar{l}) \cos(\sqrt{m^2 + |\bar{l}|^2}\tau) e^{i(m\xi + <\bar{l}\cdot\bar{\eta}>)} dm d\bar{l}.$$

As a result we have the following

Theorem 2. *Let's suppose $\varphi_0(\xi,\bar{\eta})$ is the analytical function in layer $S(\beta_0)$ ($\beta_0 = const > 0$): Fourier-image $\Phi_0(m,\bar{l}) \in \mathcal{P}_{\beta,p}$. Then the Cauchy problem (21)–(22) has the analytical solution with respect to $\xi, \bar{\eta}$ in layer $S(\beta_0)$ ($\beta_0 = const > 0$) : Fourier-image belongs to $\mathcal{P}_{\beta,p}$.*

In particular, we deduce from here that the solution of the Cauchy problem (21)–(22) is uniformly bounded with respect to τ.

Similar statement is valid for the problem (24)–(25):

Theorem 3. *Let's suppose $\varphi_k, \psi_k(\xi,\bar{\eta}), k = \pm 1$, are the analytical functions in layer $S(\beta_0)$ ($\beta_0 = const > 0$): Fourier-images belong to $\mathcal{P}_{\beta,p}$. There exist values $\theta_0,\ \beta \in (0,\beta_0) : \forall\ \theta \in [0,\theta_0]$ the Cauchy problem (24)–(25) has a unique solution, which is analytical with respect to $\xi, \bar{\eta}$ in layer $S(\beta)$: Fourier-image belongs to $\mathcal{P}_{\beta,p}$.*

The proof is based on the Fourier transformation with respect to spatial variables $\xi, \bar{\eta}$ and completely similar to [9, 10].

The analyticity of solution of (26) for the amplitudes of the first correction results from the analyticity of the right-hand side of this equation. However we will later need estimates of Fourier-images. Therefore the following statement will be useful:

Lemma 1. *Let's suppose the function $u = u(\xi,\bar{\eta},\tau)$, satisfies to the equation:*

$$[\partial_\tau + \zeta\partial_\xi]u = v(\sigma_+,\bar{\eta})w(\sigma_-,\bar{\eta}), \quad \sigma_\pm = \xi \mp \zeta\tau, \quad \zeta = const.$$

Then the following estimate is valid for Fourier-image $U = U(m,\bar{l},\tau)$:

$$\| U \|_p \leq M_0 \| V \|_p \cdot \| W \|_p, \quad M_0 = const > 0,\ \forall\ V, W \in \mathcal{P}_{\beta,p}.$$

Proof. Fourier-image $U = U(m,\bar{l},\tau)$ can be written out in an explicit form (sign $\star$ denotes convolution with respect to $(m,\bar{l})$) :

$$U = V \star \left(W \frac{\sin m\zeta\tau}{m\zeta} \right) = \int\limits_{\mathbb{R}^n} V(m - m_1, \bar{l} - \bar{l}_1) W(m_1,\bar{l}_1) \frac{\sin m_1\zeta\tau}{m_1\zeta} dm_1 d\bar{l}_1.$$

Let's divide this convolution integral by sum J_1 and J_2 with integration due to domains $\Omega_1 = \{|m_1| > 1,\ \bar{l}_1 \in \mathbb{R}^{n-1}\}$, $\Omega_2 = \{|m_1| \leq 1,\ \bar{l}_1 \in \mathbb{R}^{n-1}\}$ accordingly. In what follows we set for being brief: $\rho(m,\bar{l}) = (1+|m|+|\bar{l}|)^p e^{\beta(1+|m|+|\bar{l}|)}$.

Using the norm of space $\mathcal{P}_{\beta,p}$, we estimate integral J_1:

$$|J_1| \leq |\zeta|^{-1} \int\limits_{\mathbb{R}^n} |V(m - m_1, \bar{l} - \bar{l}_1)||W(m_1,\bar{l}_1)| dm_1 d\bar{l}_1 \leq$$

$$\leq |\zeta|^{-1} \parallel V \parallel_p \parallel W \parallel_p \int\limits_{\mathbb{R}^n} \rho^{-1}(m-m_1,\bar{l}-\bar{l}_1)\rho^{-1}(m_1,\bar{l}_1)dm_1 d\bar{l}_1 \leq$$

$$\leq M_1|\zeta|^{-1} \parallel V \parallel_p \parallel W \parallel_p \rho^{-1}(m,\bar{l}), \quad M_1 = \mathrm{const} > 0.$$

Let's estimate integral J_2:

$$|J_2| \leq \int\limits_{\Omega_2} |V(m-m_1,\bar{l}-\bar{l}_1)||W(m_1,\bar{l}_1)||\frac{\sin m_1\zeta\tau}{m_1\zeta}|dm_1 d\bar{l}_1 \leq$$

$$\leq \parallel V \parallel_p \parallel W \parallel_p \int\limits_{\Omega_2} \rho^{-1}(m-m_1,\bar{l}-\bar{l}_1)\rho^{-1}(m_1,\bar{l}_1)|\frac{\sin m_1\zeta\tau}{m_1\zeta}|dm_1 d\bar{l}_1 \leq$$

$$\leq M_2 \parallel V \parallel_p \parallel W \parallel_p \rho^{-1}(m,\bar{l}) \int\limits_0^1 \frac{\sin m_1\zeta\tau}{m_1\zeta} dm_1 \leq$$

$$\leq \pi M_2 |\zeta|^{-1} \parallel V \parallel_p \parallel W \parallel_p \rho^{-1}(m,\bar{l}), \quad M_2 = \mathrm{const} > 0.$$

The result of the obtained estimations is the validity of lemma 1.

4 Justification of the asymptotics

In this section we aim to justify an asymptotic expansion. We must prove that the constructed segment of series (11) gives an asymptotic expansion at $\varepsilon \to 0$ for some solution of the input problem (1)–(2) for long times $0 \leq t \leq T\varepsilon^{-2}$.

The proof of the theorem 1. Let's seek for the exact solution of the problem (1)–(2) in the form of:

$$u(x,\bar{y},t,\varepsilon) = \varepsilon\hat{v} + \varepsilon^2 z(x,\xi,\bar{\eta},t,\varepsilon),$$

where the function $\hat{v} = \overset{0}{v} + \varepsilon \overset{1}{v} + \varepsilon^2 \overset{2}{v}$ was constructed in section 2.

Substituting this expansion into equations (9), (10) we shall obtain the following Cauchy problem for the function $z(x,\xi,\bar{\eta},t,\varepsilon)$:

$$[\partial_t^2 - (\partial_x + \varepsilon\partial_\xi)^2 - \varepsilon^2\Delta_{\bar{\eta}} + \nu(\partial_x+\varepsilon\partial_\xi)^4 + \nu\varepsilon^4\partial_{\bar{\eta}}^4]z = \varepsilon^2 f, \tag{31}$$

$$\begin{bmatrix} z \\ \partial_t z \end{bmatrix}_{|t=0} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}(x,\xi,\bar{\eta},\varepsilon). \tag{32}$$

Here

$$f = f_0 + D_x f_1 + \varepsilon < \nabla_{\bar{\eta}} \cdot \bar{f} >, \quad \bar{f} = (f_2,\ldots,f_n),$$

$$f_0 = f_0(x,\xi,\bar{\eta},t,\varepsilon), \quad f_j = f_j(z_x, z_\xi, \nabla_{\bar{\eta}}z, x,\xi,\bar{\eta},t,\varepsilon),\ j = \overline{1,n};$$

we select the known function:

$$f_0 = 2[\partial_t\partial_\tau - \partial_x\partial_\xi + 2\nu\partial_x^3\partial_\xi] \overset{2}{v} + [\partial_\tau^2 + 2\partial_t\partial_\theta - \Delta_{\xi\bar{\eta}} + 6\nu\partial_x^2\partial_\xi^2](\overset{1}{v} + \varepsilon \overset{2}{v}) +$$

$$+2[\partial_\tau\partial_\theta + 2\nu\partial_x\partial_\xi^3]\hat{v} + \varepsilon[\partial_\theta^2 + \nu\partial_\xi^4 + \partial_{\bar{\eta}}^4](\hat{v} - v_0 + \varepsilon \overset{1}{v} + \varepsilon^2 \overset{2}{v}) + g_1^{3,0}\partial_\xi(\overset{0}{v}_x)^3;$$

and functions which depend on z:

$$f_1 = g_1(D_x[\hat{v}+\varepsilon z], \nabla_{\bar{\eta}}[\hat{v}+\varepsilon z]) - g_1^{3,0}(\overset{0}{v}_x)^3 + \nu\partial_\xi^3 v_0;$$

$$f_j = g_j(D_x[\hat{v}+\varepsilon z], \nabla_{\bar{\eta}}[\hat{v}+\varepsilon z]) + \nu\partial_{\eta_j}^3 v_0;\ j = \overline{2,n}.$$

In fact we should prove solvability and estimate of the solution of the nonlinear problem (31)–(32) for residual. Note that in this problem there is present one "superfluous" spatial variable ξ. The relation with the input problem may be obtained at contraction $\xi = \varepsilon x$. A similar method (extension of dimensionality of the problem) separates periodicity with respect to x from smoothness with respect to ξ in solution.

The proof of solvability of the problem (31)–(32) is based on the Fourier transformation with respect to variables $x, \xi, \bar{\eta}$. Thus we pass from the partial differential equation to the system of ordinary differential equations for the Fourier-image of residual $Z(k, m, \bar{l}, t, \varepsilon)$:

$$[\partial_t^2 + \lambda^2]Z = \varepsilon^2 F, \tag{33}$$

with the initial conditions

$$\left[\begin{array}{c} Z \\ \partial_t Z \end{array}\right]_{|t=0} = \left[\begin{array}{c} Z_1 \\ Z_2 \end{array}\right](k, m, \bar{l}, \varepsilon). \tag{34}$$

Here $\lambda^2 = (k+\varepsilon m)^2 + \varepsilon^2|\bar{l}|^2 + \nu(k+\varepsilon m)^4 + \nu\varepsilon^4\bar{l}^4$; $\bar{l} = (l_2, \dots, l_n)$, $\bar{l}^4 = l_2^4 + \dots + l_n^4$; capital letters are used for Fourier-images:

$$Z(k, m, \bar{l}) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} z(x, \xi, \bar{\eta}) e^{-i(kx + m\xi + <\bar{l}\cdot\bar{\eta}>)} d\xi d\bar{\eta}.$$

The right-hand side F in (33) consists of three addends:

$$\begin{aligned} F = F_0(k, m, \bar{l}, t, \varepsilon) \quad & +i(k+\varepsilon m)F_1(kZ, mZ, \bar{l}Z, k, m, \bar{l}, t, \varepsilon) + \\ & +i\varepsilon < \bar{l}\cdot\bar{F}(kZ, mZ, \bar{l}Z, k, m, \bar{l}, t, \varepsilon) >, \end{aligned}$$

and contains various convolutions of functions $kZ, mZ, \bar{l}Z$ and some known ones. In what follows sign $\star$ denotes convolution with respect to $(k, m, \bar{l})$.

Solution $Z_0 = Z_0(k, m, \bar{l}, t, \varepsilon)$ corresponding to the known right-hand side F_0 can be written out in an explicit form:

$$[\partial_t^2 + \lambda^2]Z_0 = \varepsilon^2 F_0. \tag{35}$$

We choose the solution of (35) which satisfies to the following initial conditions:

$$\left[\begin{array}{c} Z_0 \\ \partial_t Z_0 \end{array}\right]_{|t=0} = \left[\begin{array}{c} Z_1 \\ Z_2 \end{array}\right](k, m, \bar{l}, \varepsilon). \tag{36}$$

Let's seek for the solution of the problem (33)–(34) in the form of: $Z = H + Z_0$. Then we shall obtain the following Cauchy problem for H:

$$[\partial_t^2 + \lambda^2]H = i(k+\varepsilon m)F_1 + i\varepsilon < \bar{l}\cdot\bar{F} >, \tag{37}$$

$$\left[\begin{array}{c} H \\ \partial_t H \end{array}\right]_{|t=0} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]. \tag{38}$$

The linear term with the factor λ^2 in the left-hand side of (37) may be excluded by replacement of the function using fundamental solution of the corresponding homogeneous one, namely:

$$H(k, m, \bar{l}, t, \varepsilon) = Ae^{-i\lambda t} + Be^{i\lambda t},$$

where $A, B = A, B(k, m, \bar{l}, t, \varepsilon)$.

$$\text{If} \quad \mathbf{R} = \left[\begin{array}{c} A \\ B \end{array}\right], \quad \mathbf{G} = \left[\begin{array}{c} G^+ \\ G^- \end{array}\right], \quad \text{where} \quad G^\pm = \mp\frac{\Big((k+\varepsilon m)F_1 + \varepsilon < \bar{l}\cdot\bar{F} >\Big)}{2\lambda} e^{\pm i\lambda t},$$

we obtain the following differential equation for vector function $\mathbf{R}$:

$$\mathbf{R}_t = \varepsilon^2 \mathbf{G}. \tag{39}$$

The initial condition for (39):

$$\mathbf{R}|_{t=0} = 0. \tag{40}$$

After all the problem (39)–(40) is reduced to the integral equation for $\mathbf{R}(k, m, \bar{l}, t, \varepsilon)$:

$$\mathbf{R}(k, m, \bar{l}, t, \varepsilon) = \varepsilon^2 \int_0^t \mathbf{G}(k, m, \bar{l}, \mu, \mathbf{R}(k, m, \bar{l}, \mu, \varepsilon), \varepsilon) d\mu. \tag{41}$$

Here we use a scale of Banach spaces for the proof of the existence of solution, analogously [5, 9, 10, 13]. Their elements are the functions with the band of analyticity $\beta(t) > 0$. This band is narrowed down in due course. The corresponding norms are defined through the Fourier-images with exponential weights.

Definition 2. Here we introduce Banach spaces $\mathcal{H}_{\beta,p}$ of functions $U(k,m,\bar{l})$ with the finite norm

$$\| U \|_{\beta,p} = \sup_{k,m,\bar{l}} \Big[(1+|k|+|m|+|\bar{l}|)^p e^{\beta(1+|k|+|m|+|\bar{l}|)} |U(k,m,\bar{l})| \Big], \quad p \geq (n+2),\ \beta > 0.$$

Lemma 2. *The convolution operator is a bounded one in $\mathcal{H}_{\beta,p}$:*

$$\| U \star V \|_{\beta,p} \leq M_0 \| U \|_{\beta,p} \cdot \| V \|_{\beta,p}, \quad M_0 = const > 0, \quad \forall \quad U, V \in \mathcal{H}_{\beta,p}.$$

Lemma 3. *The convolution operator is a Lipschitzian one in $\mathcal{H}_{\beta,p}$: $\forall M_1 < \infty$*

$$\| U \star U - V \star V \|_{\beta,p} \leq 2 M_0 M_1 \| U - V \|_{\beta,p},$$

$$\forall \quad U, V: \quad \| U \|_{\beta,p}, \| V \|_{\beta,p} \leq M_1.$$

The proof of these statements may be obtained analogously [9, 10].

The results of it are the boundedness and Lipschitzian of any degrees of convolution in $\mathcal{H}_{\beta,p}$. We shall define a degree of convolution by relation $(\star U)^1 = U, \quad (\star U)^j = (\star U)^{j-1} \star U \ (j = 2, 3, \ldots)$.

Lemma 4. *The degree of convolution is a bounded and Lipschitzian operator in $\mathcal{H}_{\beta,p}$: $\forall M_1 < \infty$, $\forall j = 1, 2, \ldots$*

$$\|(\star U)^j\|_{\beta,p} \leq M_0^{j-1} \|U\|_{\beta,p};$$

$$\|(\star U)^j - (\star V)^j\|_{\beta,p} \leq j (M_0 M_1)^{j-1} \|U - V\|_{\beta,p};$$

$$\forall \quad U, V: \quad \| U \|_{\beta,p}, \| V \|_{\beta,p} \leq M_1, \quad M_0 = const > 0.$$

The proof is given by induction with respect to j.

We introduce the space $C([0, T\varepsilon^{-2}] \times [0, \varepsilon_0]; \mathcal{H}_{\beta,p})$ of functions $U(t,\varepsilon)$ which are continuous with respect to $(t,\varepsilon) \in [0, T\varepsilon^{-2}] \times [0, \varepsilon_0]$ with the finite norm:

$$\| U \| = \sup_{t \in [0, T\varepsilon^{-2}]} \sup_{\varepsilon \in [0, \varepsilon_0]} \| U \|_{\beta,p} .$$

Lemma 5. *The solution $Z_0(k,m,\bar{l},t,\varepsilon)$ of the Cauchy problem (35)–(36) belongs to $C([0, T\varepsilon^{-2}] \times [0, \varepsilon_0]; \mathcal{H}_{\beta,p})$ provided theorems 2 and 3.*

Proof. We can write out the solution of ordinary differential equations (35) with initial conditions (36) in an explicit form, but here it is not necessary. It is worth mentioning, that the right-hand side F_0 of this equation contains Fourier-images constructed above FAS, hence, $F_0 \in C([0, T\varepsilon^{-2}] \times [0, \varepsilon_0]; \mathcal{H}_{\beta,p})$. As all addends in F_0 have the form $\tilde{z}(k,m,\bar{L},\varepsilon)\exp(i\omega t)$ with $\omega \neq 0$, where $\tilde{z}(k,m,\bar{L},\varepsilon) \in C([0, T\varepsilon^{-2}] \times [0, \varepsilon_0]; \mathcal{H}_{\beta,p})$, the solution $Z_0(k,m,\bar{l},t,\varepsilon) \in C([0, T\varepsilon^{-2}] \times [0, \varepsilon_0]; \mathcal{H}_{\beta,p})$. Lemma 5 is proved.

In what follows we introduce Banach spaces $\mathbf{H}_{\beta,p}$ of vector functions $\mathbf{R}(k,m,\bar{l},t,\varepsilon)$ with continuous components with respect to $k, m, \bar{l}, t, \varepsilon$ and with exponential decreasing at infinity with respect to $k, m, \bar{l}$. The norm in space $\mathbf{H}_{\beta,p}$ is defined as the sum of norms of the components of vector $\mathbf{R}$ in space $\mathcal{H}_{\beta,p}$:

$$\| \mathbf{R} \|_{\beta,p} = \| A \|_{\beta,p} + \| B \|_{\beta,p} .$$

Vector functions $\mathbf{R}(k,m,\bar{l},t,\varepsilon) \in \mathbf{H}_{\beta,p}$, which are continuously depending with respect to $t \in [0, T\varepsilon^{-2}], \varepsilon \in [0, \varepsilon_0]$ are considered in Banach space $\mathbf{C}_p = C([0, T\varepsilon^{-2}] \times [0, \varepsilon_0]; \mathbf{H}_{\beta,p})$ with the norm:

$$\| \mathbf{R} \|_{\mathbf{C}_p} = \sup_{t,\varepsilon} \| \mathbf{R}(k,t,m,\bar{l},\varepsilon) \|_{\beta,p} .$$

In what follows the index $\beta = \beta(\theta)$ will depend on $\theta = \varepsilon^2 t$.

The considered equations reduce to integration operator from (41), which contains both convolution degrees and exterior factors $(k + \varepsilon m)/(2\lambda)$; $\varepsilon l_j/(2\lambda)$, which are uniformly bounded (the majorant is 1/2).

Factors k, m, l_j coming from operators ∂_x, ∂_ξ, ∂_{η_j} are unbounded ones in $\mathbf{H}_{\beta,p}$. But they will be bounded from $\mathbf{H}_{\beta,p}$ in $\mathbf{H}_{\beta,p-1}$. On the other hand, the integration operator

$$I[\mathbf{R}] = \varepsilon^2 \int_0^t \mathbf{R}(k,m,\bar{L},\mu,\varepsilon)d\mu$$

is linear bounded from $\mathbf{C}_{p-1}$ in $\mathbf{C}_p$ with $\beta = \beta_0 - \beta_1\varepsilon^2 t$ (β_0,$\beta_1 = \text{const} > 0$) on the functions continuously depending with respect to t,ε. This integration operator has estimate analogously [9, 10]:

$$\parallel I[\mathbf{R}] \parallel_{\mathbf{C}_p} \leq \beta_1^{-1} \parallel \mathbf{R} \parallel_{\mathbf{C}_{p-1}}, \quad \forall \quad t \leq \beta_0\beta_1^{-1}\varepsilon^{-2}.$$

If now we choose a sufficiently large constant β_1, the composition $I[(k+m+l_j)\mathbf{R}]$ operator of multiplication with respect to $(k+m+l_j)$ with the integrated one will be the contractive operator in scale of spaces $\mathbf{H}_{\beta,p}$ on functions, continuously depending with respect to t,ε. A similar situation is in case with general operator from (41). Integrals in the right-hand side (41) will be bounded and Lipschitzian operators in $\mathbf{C}_p$. Therefore the integration operator from (41) is contractive in $\mathbf{C}_p$ with $\beta = \beta_0 - \beta_1\varepsilon^2 t$ (β_0,$\beta_1 = \text{const} > 0$). This is what ensures a local solvability of integral equations (41) in $\mathbf{C}_p$. Thus the upper bound of existence interval $T \leq \beta_0\beta_1^{-1}\varepsilon^{-2}$ is defined from the condition $\beta = \beta_0 - \beta_1\varepsilon^2 t \geq 0$ ($\beta_0, \beta_1 = \text{const} > 0$). Thus the leading term of the asymptotics in (11) is justified. Theorem 1 is proved.

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